A Nim game played on graphs

Masahiko Fukuyama
Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan
e-mail: fuquyama@interlink.or.jp

Abstract We propose a new impartial game played by two players, which can be compared to the well-known Nim game [1, 3, 4] played on graphs. In this paper, we consider this game and investigate its winning strategies. In the proof, Menger's theorem [2] noted in graph theory plays a crucial role.

Keywords: Impartial game; Nim; Menger's theorem; Grundy number;

1 Rule of Nim on graphs

A variety of Nim-type games has been proposed and studied. In this paper we also propose a new one, as it were, Nim game played on graphs. So we call this game *Nim on graphs.*

The rule of Nim on graphs is as follows. At first, to set a starting position of the game, we fix some finite undirected graph and assign to each edge a non-negative integer. Further we take one piece and put it at a vertex of the graph. From this given position, the game starts and proceeds by the two players' alternate moves with the following series of choices.

(i) Choose an edge incident with the vertex of the piece.

(ii) Decrease the value of this edge to any non-negative integer strictly.

(iii) Move the piece to the adjacent vertex along this edge.

The game ends when a player in his turn can not move since the value of each edge incident with the piece's vertex is equal to zero. Then, according to the normal play convention, this player is taken as the loser.

We remark that the ordinary Nim is a special case of our game. The ordinary Nim with N heaps of sizes $m_1, m_2, \dots, m_N$ is equivalent to the Nim on the graph which consists of two vertices and N edges joining these two vertices, as Figure 1. Here the symbol "△" indicates the piece, and $m_1, m_2, \dots, m_N$ indicates the non-negative integers assigned to the respective edges.

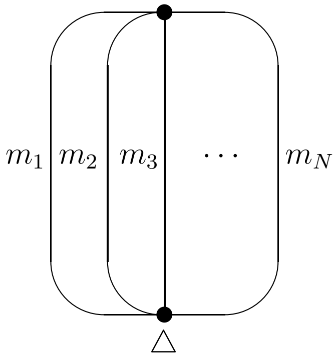
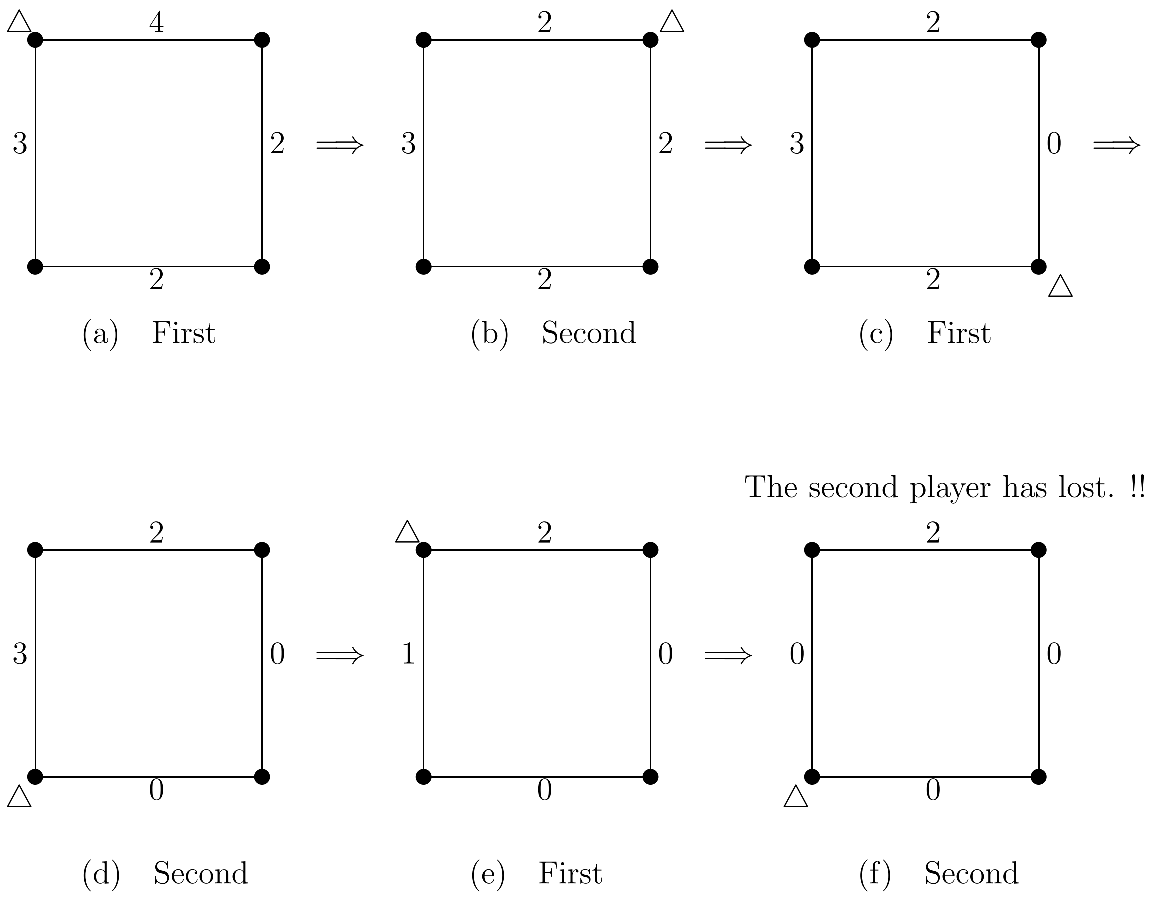


Figure 1. The Nim on graphs equivalent to the ordinary Nim

Figure 2 illustrates a transition of positions in an actual match of Nim on graphs. In the starting position(a), the first player chooses the right edge of the piece △, decrease its value from 4 to 2 and moves the piece right along this edge, which makes (a) into (b). Next, in the position(b), the second player chooses the down edge of the piece, decreases its value from 2 to 0 and moves the piece down, which makes (b) into (c). Similarly the players move alternately and the positions(c), (d), (e) and (f) result in this order. Finally, in the position(f), the second player has no moves since each edge incident with the vertex of the piece △ is assigned to zero. Then, the first player wins this match.

To tell the truth, whenever the game starts from the position(a), the first player can win for any second player's move. In other words, in the starting position(a), the first player has a winning strategy. In this paper, we are concerned with the problem whether, in the given starting position, the first player or the second player has a winning strategy. By virtue of Menger's theorem, we obtain a theorem(Theorem 3.5) which gives the solution of this problem under certain hypothesis on the structure of the graph of the starting position. This is the main result of this paper.

In Section 2, we shall introduce some notations and terminology. In Section 3, we shall present and prove the main theorem. In Section 4, we shall extend our game, which enables us to study Nim on graphs with multiple edges.



(a) First (b) Second (c) First

The second player has lost. !!

(d) Second (e) First (f) Second

Figure 2. An example of transitions of positions in an actual match

2 Definition

All considered graphs of this paper are undirected and finite. For a graph G, we denote by $V(G)$ and $E(G)$ the set of its vertices and the set of its edges, respectively. We assign to each edge $e \in E(G)$ a non-negative integer $\omega(e)$, which is called the *weight of* e, by a mapping $\omega : E(G) \longrightarrow \overline{\mathbb{N}} = \mathbb{N} \cup \{0\}$. We denote by $G_{\omega,v}$ the position of the game designated by a graph G, a weight mapping $\omega : E(G) \longrightarrow \overline{\mathbb{N}}$ and a vertex $v \in V(G)$ for the vertex of the piece. Either player may move in his turn from a position $G_{\omega,v}$ to a position $G_{\omega',v'}$ if and only if there is an edge $e \in E(G)$ joining v and v' such that $\omega'(e) < \omega(e)$ and $\omega'(f) = \omega(f)$ for any $f \neq e$. Then we call $G_{\omega',v'}$ an *option of* $G_{\omega,v}$. A player given a position without options in his turn is the loser(the normal play convention).

We assume that any graph of the game forms a bipartite graph without multiple edges, except Lemma 3.3 and Section 4. We fix some vertex v_0 of G and say that a vertex of G is *even* (respectively *odd*) if it takes even (respectively odd) steps from v_0. When the game starts from a position, if the first (respectively second) player can win for any second (respectively first) player's move, we say that this position is a *p-position* (respectively 0*-position*), which is named after the fact that its Grundy number [1, 3, 4] is positive (respectively 0). When a game starts from a position which has no options, the first player has already lost this match. So, we take this position as a 0-position. The following is a basic property for p-positions and 0-positions.

(i) A position is a p-position if and only if it has a 0-position option.

(ii) A position is a 0-position if and only if it has no 0-position options.

Furthermore, we assume that

$$\text{the degree of any odd vertex of the graph of the game is just two,} \tag{2.1}$$

except Lemma 3.3 and Section 4. As mentioned above, we are concerned with the following problem under this hypothesis (2.1).

Problem A : Find whether the given position is a p-position or a 0-position.

In this paper, we use the term *path of length* N as a graph which has $V(G)$ and $E(G)$ with the form $V(G) = \{v_0, v_1, \ldots, v_N\}$ and $E(G) = \{v_0v_1, v_1v_2, \ldots, v_{N-1}v_N\}$, respectively. That is, any path does not encounter the same vertex twice. We denote this path by $v_1v_2 \cdots v_N$.

Definition 2.1. For a weighted graph G_ω, a path of even length $L = v_0v_1 \cdots v_{2N}$ included in G such that

$$\omega(v_{2N-1\,2N}) = 0 \quad \text{and} \quad \omega(v_{j\,j+1}) > 0 \quad \text{for } 0 \leqq j < 2N-1 \tag{2.2}$$

is called an *odd path starting from* v_0 *of* G_ω. Then the vertex v_{2N-1} is called the *terminal vertex of this odd path.* Especially, for a position with the piece at even vertex v, an odd path starting from the piece's vertex v is simply called an *odd path of this position.* By the following proposition, let us agree to call a position which has at least one odd path a *trivial p-position.*

Proposition 2.1. A position which has at least one odd path is a p-position.

Proof. Note that the piece is at even vertex. When the game starts from a position with an odd path L, the first player can always win by the strategy to move the piece along L toward the terminal vertex of L and to decrease the weight of the chosen edge to 0 in his turn. By this strategy, he will certainly defeat his opponent, because the second player cannot but move the piece along L toward the terminal vertex of L and inevitably loses any option when the piece arrives at the terminal vertex of L. □

Remark 2.1. We should notice that, in any trivial p-position, the piece is at even vertex. So the starting vertex of each odd path of trivial p-positions is always even and its terminal vertex is always odd.

Example 2.1. The position in Figure 3 is an example of a trivial p-position. In the figures below, the drawn circles “$\bullet$” usually indicate the even vertices, the outlined circles ”$\circ$” the odd vertices and the symbol “$\triangle$” the piece of the position. The position in Figure 3 has an odd path, which is indicated by the thick line.

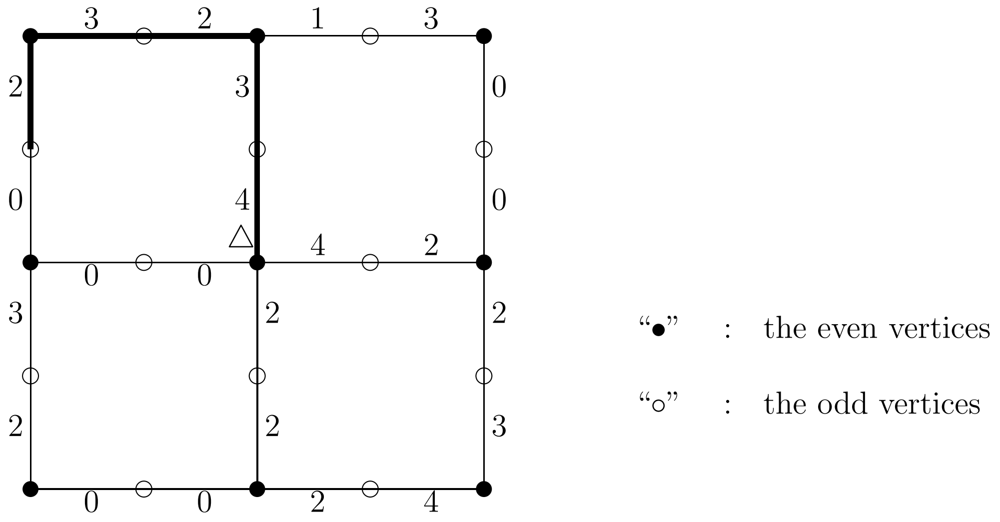


Figure 3. A trivial p-position
The thick line indicates an odd path of this position.

Definition 2.2. Let $\tilde{G}$ be a subgraph of G. For weighted graphs G_ω and $\tilde{G}_{\tilde{\omega}}$, we let $G_{\omega+\tilde{\omega}}$ denote the superposition of them, which is defined as the weighted graph with the weight mapping $\omega+\tilde{\omega}$ given by

$$(\omega+\tilde{\omega})(e) = \begin{cases} \omega(e)+\tilde{\omega}(e) & \text{for } e \in E(\tilde{G}) \\ \omega(e) & \text{for } e \in E(G) \setminus E(\tilde{G}). \end{cases} \tag{2.3}$$

Further, let us define the superposition of a position $G_{\omega,v}$ and a weighted graph $\tilde{G}_{\tilde{\omega}}$ as the position $G_{\omega+\tilde{\omega},v}$.

If G_ω can be regarded as the superposition of $\tilde{G}_{\tilde{\omega}}$ and some weighted graph, we say that G_ω includes $\tilde{G}_{\tilde{\omega}}$. We also say that $G_{\omega,v}$ includes $\tilde{G}_{\tilde{\omega}}$ if G_ω includes $\tilde{G}_{\tilde{\omega}}$.

3 Main result for Nim on bipartite graphs

Now we proceed to our main theorem and its proof. In the lemmas or the propositions below, we shall often use induction on options of the position.

In this paper, we use the term *cycle of length* N as a graph G which has $V(G)$ and $E(G)$ with the form $V(G) = \{v_1, v_2, \ldots, v_N\}$ and $E(G) = \{v_1v_2, v_2v_3, \ldots, v_Nv_1\}$, respectively. That is, the degree of any vertex of a cycle is always two. We denote by G_1 a weighted graph with the weight mapping 1, which is defined as the weight mapping assigning the weight 1 to each edge of G. When a graph G forms a path (respectively a cycle), we call G_1 a 1-*path* (respectively 1-*cycle*).

Lemma 3.1. Let G be a graph and C a cycle. Suppose that C is a subgraph of G.
(i) The superposition of a p-position $G_{\omega,v}$ with the piece at even vertex v and a 1-cycle C_1 is a p-position.
(ii) The superposition of a 0-position $G_{\omega,v}$ with the piece at odd vertex v and a 1-cycle C_1 is a 0-position.

Proof. We let $G_{\omega+1_C,v}$ denote the superposition of $G_{\omega,v}$ and C_1. We shall use induction on options of the position.

(i) Since $G_{\omega,v}$ is a p-position with the piece at even vertex, $G_{\omega,v}$ has a 0-position option $G_{\omega',v'}$ with the piece at odd vertex. By induction and applying (ii) to $G_{\omega',v'}$, the superposition of $G_{\omega',v'}$ and C_1 is a 0-position. Since this superposition is an option of $G_{\omega+1_C,v}$, the position $G_{\omega+1_C,v}$ is a p-position.

(ii) We shall divide its proof into two cases.

The case when $G_{\omega,v}$ has an option Since $G_{\omega,v}$ is a 0-position with the piece at odd vertex, each option $G_{\omega',v'}$ of $G_{\omega,v}$ is a p-position with the piece at even vertex. We should notice that the superposition of $G_{\omega',v'}$ and a 1-cycle is a p-position, which follows from induction and applying (i) to $G_{\omega',v'}$.

If v is not in C, each option of $G_{\omega+1_C,v}$ can be regarded as the superposition of an option of $G_{\omega,v}$ and C_1 and, so it is a p-position.

On the other hand, if v is in C, we observe that each option of $G_{\omega+1_C,v}$ which results by decreasing the weight of the chosen edge to 0 is a trivial p-position, and that any other option of $G_{\omega+1_C,v}$ can be regarded as the superposition of an option of $G_{\omega,v}$ and C_1 and, so it is also a p-position.

Consequently, each option of $G_{\omega+1_C,v}$ is a p-position in any case. Therefore, $G_{\omega+1_C,v}$ is a 0-position.

The case when $G_{\omega,v}$ has no options If v is not in C, $G_{\omega+1_C,v}$ has also no options. If v is in C, we see that each option of $G_{\omega+1_C,v}$ is a trivial p-position. Therefore, $G_{\omega+1_C,v}$ is a 0-position. □

Lemma 3.2. A position with the piece at even vertex is a p-position if and only if this position can be regarded as the superposition of a trivial p-position and 1-cycles.

Proof. It follows from Lemma 3.1 that the superposition of a trivial p-position and 1-cycles is a p-position. So it remains only to show that a p-position with the piece at even vertex can be regarded as the superposition of a trivial p-position and 1-cycles. We let $G_{\omega,v}$ be a non-trivial p-position with the piece at even vertex v and let $G_{\omega',v'}$ be a 0-position option of $G_{\omega,v}$. Let u be the vertex which is adjacent to the odd vertex v' and not v, as Figure 4. We denote the edges $v'v$ and $v'u$ by e_v and e_u, respectively. Let us take the p-position option $G_{\omega'',v''}$ of $G_{\omega',v'}$ according to the following list.

(i) If $\omega'(e_v) > \omega'(e_u)$,
let $v'' = v$ and $\omega''(e_v) = \omega'(e_u)(= \omega''(e_u))$.

(ii) If $\omega'(e_v) < \omega'(e_u)$,
let $v'' = u$ and $\omega''(e_u) = \omega'(e_v)(= \omega''(e_v))$.

(iii) If $\omega'(e_v) = \omega'(e_u)$,
let $v'' = u$ and
$\omega''(e_u) = \omega'(e_u) - 1(= \omega''(e_v) - 1)$.

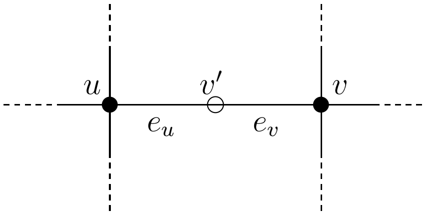


Figure 4.

By induction and applying this lemma to this p-position $G_{\omega'',v''}$, it can be regarded as the superposition of a trivial p-position, which is denoted by $G_{\zeta'',v''}$, and 1-cycles. We let L'' denote an odd path of $G_{\zeta'',v''}$ and let $G_{\zeta,v}$ denote the position obtained by adding the difference between G_ω and $G_{\omega''}$ to the weighted graph $G_{\zeta''}$, that is,

$$\zeta(e) = \zeta''(e) + (\omega(e) - \omega''(e)) \quad \text{for each } e \in E(G)$$

and putting the piece at v. Then the position $G_{\omega,v}$ is the superposition of $G_{\zeta,v}$ and 1-cycles. So it suffices to show that $G_{\zeta,v}$ is a trivial p-position, or that $G_{\zeta,v}$ has at least one odd path. In the respective cases (i)-(iii), we shall furthermore divide the argument into the four cases below.

(a) The case when L'' goes through neither of the edges e_v nor e_u

(b) The case when L'' goes through both of the edges e_v and e_u

(c) The case when L'' goes through the edge e_v and not through the edge e_u

(d) The case when L'' goes through the edge e_u and not through the edge e_v

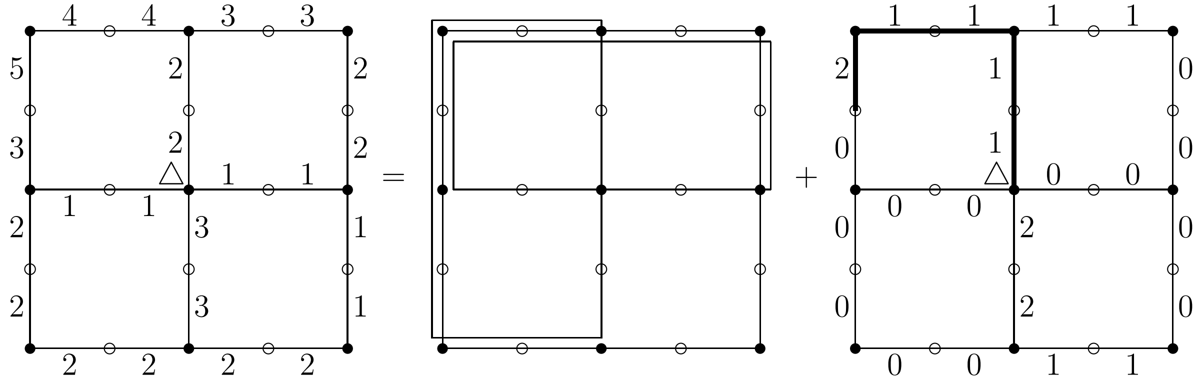
The case when $G_{\omega'',v''}$ is taken according to (i) In the case (a) or (b), L'' is an odd path not only of $G_{\zeta'',v''}$ but also of $G_{\zeta,v}$. Then, in either case, $G_{\zeta,v}$ is a trivial p-position. On the other hand, both the cases (c) and (d) are impossible because $\omega''(e_v) = \omega''(e_u)$.

The case when $G_{\omega'',v''}$ is taken according to (ii) In the case (a), we can find an odd path of $G_{\zeta,v}$ by adding the path $vv'u$ to L''. In the case (b), we can find one by removing the path $vv'u$ from L''. Then, in either case, $G_{\zeta,v}$ is a trivial p-position. On the other hand, both the cases (c) and (d) are impossible because $\omega''(e_u) = \omega''(e_v)$.

The case when $G_{\omega'',v''}$ is taken according to (iii) In the case (a) or (b), we find an odd path of $G_{\zeta,v}$ in the same way as above. The case (d) is impossible because $\omega''(e_u) < \omega''(e_v)$. In the case (c), noting that $G_{\omega'',v''}$ is taken according to the case (iii), it is easily seen that the weight of the edge e_u of $G_{\zeta,v}$ is just equal to 1 and that of the edge e_v of $G_{\zeta,v}$ is greater than or equal to 2. So we have just one cycle in $G_{\zeta,v}$ which goes through both e_u and e_v. We remove this 1-cycle from $G_{\zeta,v}$ and denote the remainder by $G_{\zeta,v}$ afresh. Then, since $\zeta(v'v) > 0$ and $\zeta(v'u) = 0$, we find an odd path $vv'u$ of $G_{\zeta,v}$.

Consequently we conclude that $G_{\omega,v}$ can be regarded as the superposition of a trivial p-position and 1-cycles. □

Example 3.1. We illustrate Lemma 3.2 with the position(a) given in Figure 5. We observe that the position(a) can be regarded as the superposition of three 1-cycles (Figure 5(b)) and a trivial p-position (Figure 5(c)) with an odd path indicated by the thick line. Thus, it follows from Lemma 3.2 that the position(a) is a p-position. Here the addition "+" stands for the superposition.



(a) A p-position (b) Three 1-cycles (c) A trivial p-position

Figure 5. A position(a) which can be regarded as the superposition of a trivial p-position(c) and 1-cycles (b).

To state the next lemma, we introduce some terminology. Take a graph G and let u and v be distinct two vertices of G. Then, we call an edge set $E \subset E(G)$ a *cut separating u and v* if every path connecting u and v includes an edge of E. For a weighted graph G_ω and a cut E of G, we call the sum of $\omega(e)$ over $e \in E$ the *capacity of this cut E*. For a weighted graph G_ω and two distinct vertices u and v, we call a cut separating u and v which minimizes its capacity a *minimum cut separating* u *and* v.

In the next lemma, we do not assume that G is bipartite, and G may have loops or multiple edges.

Lemma 3.3. Take a weighted graph G_ω. Let u and v be distinct two vertices of G. The minimum capacity of cuts separating u and v of G_ω is equal to the maximum number of 1-paths included in G_ω which connect u and v.

Proof. Replace each edge e of G by $\omega(e)$ edges joining the same endvertices as e. Let $\overline{G}$ be the graph obtained by this replacing; see Figure 6. Note that 1-paths included in G_ω correspond to edge-disjoint paths of $\overline{G}$ and that the minimum capacity of cuts separating u and v of G_ω corresponds to the minimum number of edges separating u and v of $\overline{G}$. Applying the edge form of Menger's theorem [2] to this graph $\overline{G}$ proves this lemma. □

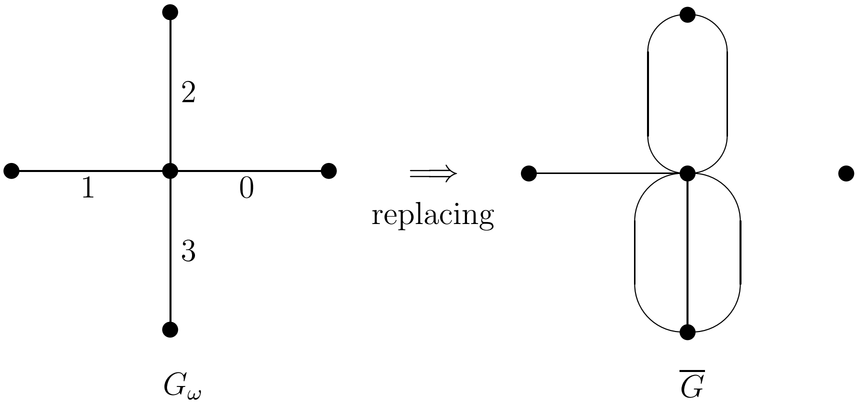


Figure 6. A weighted graph G_ω and the graph $\overline{G}$ obtained by the replacing in the proof of Lemma 3.3

Definition 3.1. For a weighted graph G_ω and an odd vertex u of G, suppose that the weights of the two edges incident with u are different from each other. Then, of the two edges, the edge with the larger (respectively smaller) weight is called the *thick* (respectively *thin*) *edge of* u. For the weighted graph which results by cutting off G_ω at u, we call the section incident with the thick (respectively thin) edge the *thick* (respectively *thin*) *section*; see Figure 7.

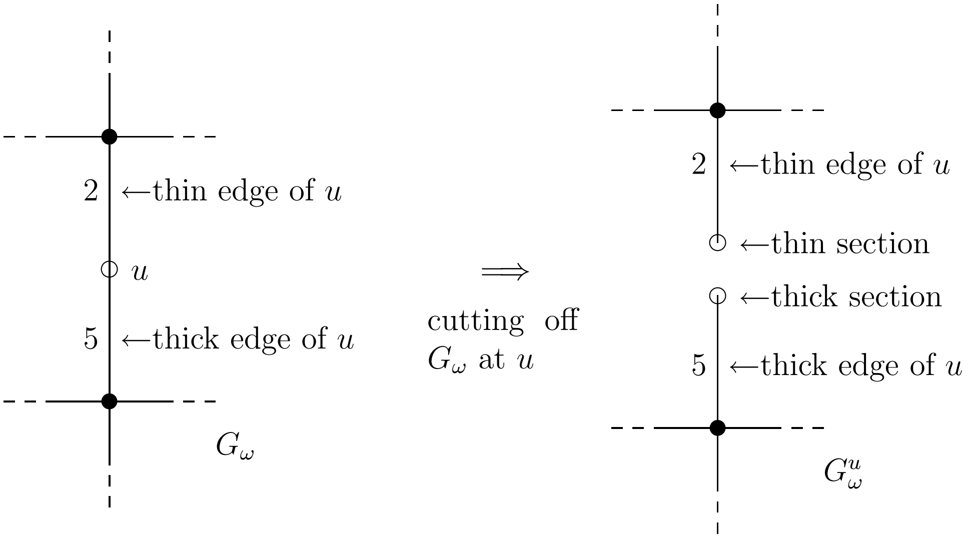


Figure 7. Cutting off G_ω at u

Lemma 3.4. Take a position $G_{\omega,v}$ with the piece at even vertex v. Let G^u_ω denote the weighted graph which results by cutting off G_ω at an odd vertex u of G.

Then, $G_{\omega,v}$ can be regarded as the superposition of 1-cycles and a trivial p-position with an odd path which terminates at u if and only if the following three conditions are satisfied.

(i) The weights of the two edges incident with u are different from each other.

(ii) The minimum capacity of cuts separating the two sections of G^u_ω is equal to the weight of the thin edge of u.

(iii) Even if any minimum cut separating the two sections is removed from the weighted graph G^u_ω, the vertex v is always connected with the thick section.

Proof. Let $m \in \overline{\mathbb{N}}$ be the weight of the thin edge of u. Let $\hat{G^u}_{\hat{\omega}}$ denote the weighted graph which results by appending to G^u_ω a new edge joining v and the thin section with weight 1.

1) Suppose that $G_{\omega,v}$ is the superposition of 1-cycles and a trivial p-position with an odd path terminating at u. This structure obviously ensures the condition (i). First, let us show that the condition (ii) is satisfied. Now, without loss of generality, we can assume that each 1-cycle goes through u because we can include any 1-cycle not going through u in the trivial p-position. So $G_{\omega,v}$ can be regarded as the superposition of one trivial p-position and just m 1-cycles going through u. Then, in G^u_ω, the maximum number of 1-paths connecting the two sections is equal to m. Therefore, by Lemma 3.3, the minimum capacity of cuts separating the two sections of G^u_ω is also equal to m, which implies that the condition (ii) is satisfied.

Next, to show that the condition (iii) is satisfied, we suppose that the vertex v and the thick section are disconnected in the weighted graph obtained by removing a minimum cut E separating the two sections from G^u_ω. Then, also in $\hat{G^u}_{\hat{\omega}}$, E is a minimum cut separating the two sections, whose capacity is equal to m. Therefore, Lemma 3.3 shows that, in $\hat{G^u}_{\hat{\omega}}$, the maximum number of 1-paths connecting the two sections is also equal to m. On the other hand, noting the structure of $G_{\omega,v}$, that is, the superposition of one trivial p-position with an odd path terminating at u and just m 1-cycles going through u, we observe that $\hat{G^u}_{\hat{\omega}}$ includes $m+1$ 1-paths connecting the two sections, which implies contradiction.

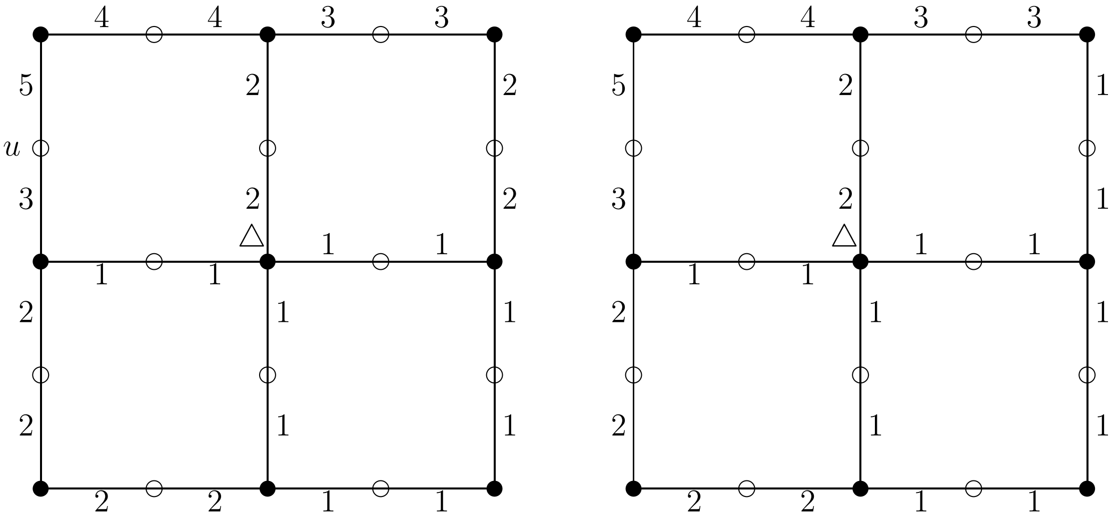
2) Suppose that $G_{\omega,v}$ has an odd vertex satisfying the conditions (i)-(iii). Then, we easily see that, in $\hat{G^u}_{\hat{\omega}}$, the minimum capacity of cuts separating the two sections is equal to $m+1$. Therefore, by Lemma 3.3, $\hat{G^u}_{\hat{\omega}}$ includes $m+1$ 1-paths connecting the two sections. This structure implies that $G_{\omega,v}$ can be regarded as the superposition of m 1-cycles and a trivial p-position with an odd path terminating at u. $\square$

The following theorem is the main theorem, which is an immediate consequence of Lemma 3.2 and Lemma 3.4.

Theorem 3.5. Let $G_{\omega,v}$ be a position with the piece at even vertex v. Then, $G_{\omega,v}$ is a p-position if and only if $G_{\omega,v}$ has an odd vertex u satisfying the conditions (i)-(iii) of Lemma 3.4.

Theorem 3.5 gives the solution of Problem A for any position of Nim on bipartite graphs satisfying the hypothesis (2.1). The following examples illustrate it.

Example 3.2. To find whether the position(a) or (b) in Figure 8 is a p-position or a 0-position, let us apply Theorem 3.5 to these positions, respectively. Since the position(a) has an odd vertex u satisfying the three conditions of Lemma 3.4, it is a p-position. On the other hand, since the position(b) has no odd vertices satisfying them, it is a 0-position.



(a) A p-position (b) A 0-position

Figure 8. Examples to which one can apply Theorem 3.5

4 Nim on graphs with multiple edges

In this section, we shall deal with Nim on graphs with multiple edges. In order to do this, we shall extend our game, Nim on graphs. This extended game, which is compared to impartial games played on graphs, serves to reduce the problem for finding the Grundy number [1, 3, 4] of Nim on graphs with multiple edges to that for one without multiple edges. In this section, we do not assume that graphs are bipartite, and graphs may have loops or multiple edges.

Extended Nim on graphs

In brief, our extended game is defined as Nim on graphs modified by assigning either a position of an impartial game [1, 3] or a weight (non-negative integer) to each edge. Let $\omega(e)$ denote either the position or the weight assigned to e.

The rule of this extended game is as follows. At first, to set a starting position of the game, we fix some finite undirected graph and assign to each edge either a position of an impartial game or a weight (non-negative integer). Further we take one piece and put it at a vertex of the graph. From this position, the game starts and proceeds by the two players' alternate moves with the following series of choices.

(i) Choose an edge e incident with the vertex of the piece.

(ii) If a weight is assigned to this edge e, decrease $\omega(e)$ to any non-negative integer strictly. If a position of an impartial game is assigned to this edge e, play one move from this position $\omega(e)$ of this impartial game.

(iii) Move the piece to the adjacent vertex along this edge e.

The game ends, when a player in his turn has no options since the game assigned to any edge incident with the piece's vertex is already ended and the weight of any edge incident with the piece's vertex is equal to zero. Then, according to the normal play convention, this player is taken as the loser. By the following proposition, the problem for finding the Grundy number of the extended Nim on graphs can be reduced to that for the normal Nim on graphs. Noting the definition of the Grundy number, one can prove this proposition easily. So, we shall omit its proof.

Proposition 4.1. Let $G_{\omega,v}$ be a position of the extended Nim on graphs. Let $g(G_{\omega,v})$ denote the Grundy number of this position $G_{\omega,v}$. When $\omega(e)$ denotes the position of the impartial game assigned to e, let $g(\omega(e))$ denote the Grundy number of this position $\omega(e)$. Take the weight mapping $\omega_0 : E(G) \to \overline{\mathbb{N}}$ given by

$$\omega_0(e) = \begin{cases} g(\omega(e)) & \text{if } \omega(e) \text{ denotes the position of the impartial game assigned to } e \\ \omega(e) & \text{if } \omega(e) \text{ denotes the weight assigned to } e. \end{cases} \tag{4.1}$$

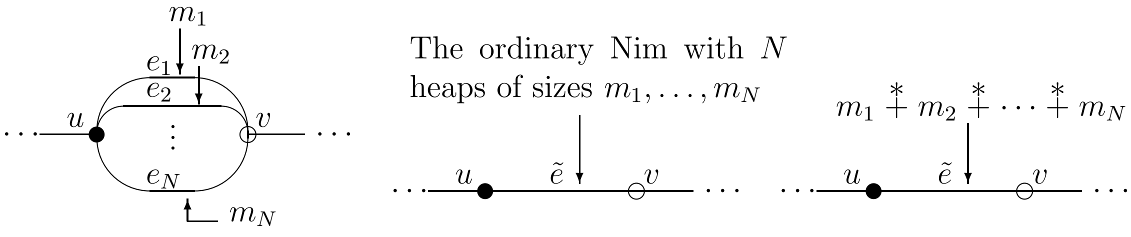
Then,

$$g(G_{\omega,v}) = g(G_{\omega_0,v}) \tag{4.2}$$

holds. Here $G_{\omega_0,v}$ is the position of the normal Nim on the graph G with the weight mapping ω_0.

Nim on graphs with multiple edges

Now we shall remark that Proposition 4.1 serves to reduce the problem for finding the Grundy number of Nim on graphs with multiple edges to that for one without multiple edges. Consider a position of Nim on graphs with a set of multiple edges $e_1, e_2, \ldots, e_N$, as Figure 9(a), where $m_1, m_2, \ldots, m_N$ are their respective weights. For this position, construct the position of the extended Nim on graphs by replacing the whole of these multiple edges with one edge $\tilde{e}$ and assigning to $\tilde{e}$ the position of the ordinary Nim with N heaps of sizes $m_1, \ldots, m_N$, as Figure 9(b). Then we should notice that these positions are equivalent to each other, and thus have the same Grundy number.



(a) (b) (c)

Figure 9. A position(a) of Nim on graphs with multiple edges and the position(c) of Nim on graphs without multiple edges which has the same Grundy number as the position(a).

Recall that the Grundy number of the ordinary Nim with N heaps of sizes $m_1, m_2, \dots, m_N$ is equal to $m_1 \overset{*}{+} m_2 \overset{*}{+} \cdots \overset{*}{+} m_N$, where the operation $\overset{*}{+}$ is the Nim sum [1, 3, 4]. Then, applying Proposition 4.1 to the position(b) shows that the Grundy number of the position(b) is equal to that of the position(c) in Figure 9 of the normal Nim on the same graph, which results by assigning to $\tilde{e}$ the weight $m_1 \overset{*}{+} m_2 \overset{*}{+} \cdots \overset{*}{+} m_N$ in place of the ordinary Nim position. Consequently, the Grundy number of the position(a) is equal to that of the position(c).

Example 4.1. The following is an example of Nim on graphs with multiple edges. Noting that $4 \overset{*}{+} 6 \overset{*}{+} 3 = 1$ and $5 \overset{*}{+} 4 = 1$, the argument above shows that the Grundy number of the position(a) in Figure 10 is equal to that of the position(b) in Figure 10, which is equal to 1 by the definition of the Grundy number [1, 3, 4].

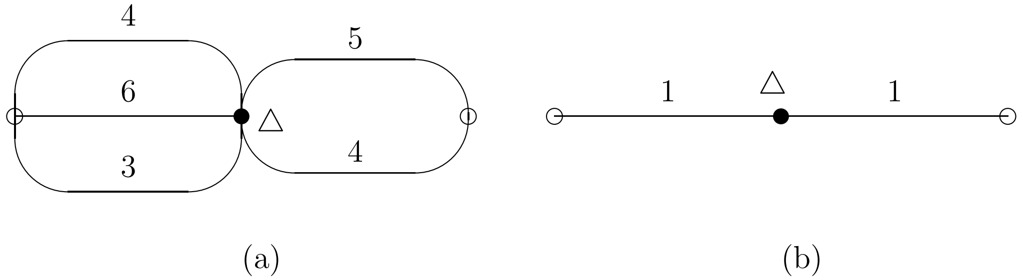


Figure 10. The Grundy number of the position(a) with multiple edges is equal to that of the position(b) without multiple edges, which is equal to 1.

Acknowledgements

I wish to thank Yôhei Yamasaki for helpful comments on this paper and Tadahisa Funaki for his encouragement and support.

References

[1] E.R. Berlekamp, J.H. Conway, R.K. Guy, Winning Ways, for Your Mathematical Plays (Academic Press, New York, 1982).

[2] B. Bollobás, Graph Theory, An Introductory Course (Springer, New York, 1979).

[3] J.H. Conway, On Numbers and Games (Academic Press, London and New York, 1976).

[4] Y. Yamasaki, Combinatorial Games: Back and Front (Springer, Tokyo, 1989)(in Japanese).